

# Symmetries of 2nd order ODE:

$$y'' + G(x)y' + H(x)y = 0.$$

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## Abstract

This paper is devoted to study the Lie algebra of linear symmetries of a homogenous 2nd order ODE, by the method of Kushner, Lychagin and Robstov [1].

**Key Words:** *linear differential equation, differential operator, symmetry.*

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## Introduction

Symmetries of differential equations make a magnificent portion in theory of differential equations, and there are so much researches in this object. Here we are going to decompose the structure of Lie algebra of linear symmetries of ODE,  $y'' + G(x)y' + H(x)y = 0$ , where  $G$  and  $H$  are smooth functions of  $x$ , to two subalgebras which are called **even** and **odd** symmetries of ODE. First the meaning of even and odd symmetries for a differential operator is given, next we will find these two concepts for the differential operator corresponding to ODE instead of the equation itself. Reader is referred to [2] and [3] for some fundamental contexts in geometry of manifolds and their applications to theory of differential equations.

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# 1 Symmetries of ODEs

Consider a general  $n$ -th order differential equation  $\Delta = 0$  which is defined on  $n$ -th jet space of  $p$  independent and  $q$  dependent variables. As we know a *symmetry* of the system of above differential equation means a point (or contact) transformation which maps solutions to solutions. In the case of point transformation, the infinitesimal generator  $\mathbf{v}$  from a Lie algebra  $\mathfrak{g}$  corresponding to group transformation makes a symmetry of  $\Delta = 0$ , if its  $n$ -th prolongation annihilate  $\Delta$ , i.e.,  $\mathbf{v}^{(n)}(\Delta) = 0$ . See [2] and [3] for more details of symmetries of differential equations. It is noteworthy that all manifolds, vector fields, differential forms and... are seem to be smooth in the sequel.

## 1.1 Generating Functions

Let us consider an ODE of  $(k+1)$ -th order which is resolved with respect to the highest derivative:  $y^{(k+1)} = F(x, y, y', \dots, y^{(k)})$ . This equation determines a one-dimensional distribution on the  $k$ -th jet space with one independent variable  $x$  with coordinate  $(x, y = p_0, p_1, \dots, p_k)$ , which is generated by the vector field

$$\mathcal{D} = \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial p_0} + \dots + p_k \frac{\partial}{\partial p_{k-1}} + F \frac{\partial}{\partial p_k},$$

or by the contact differential 1-forms

$$\omega^1 = dp_0 - p_1 dx, \dots, \quad \omega^k = dp_{k-1} - p_k dx, \quad \omega^{k+1} = dp_k - F dx.$$

Consider a vector field  $X$  on manifold  $M$ ,  $X$  is called a *symmetry* of the distribution  $P$ , if the distribution is invariant under the flow of  $X$ . Denote by  $\text{Sym}(P)$  the set of all symmetries of  $P$ . If  $X$  belongs to  $P$  then it is called a *characteristic symmetry* and the set of all characteristic symmetries of  $X$  is denoted by  $\text{Char}(P)$  which makes an ideal of  $\text{Sym}(P)$ .

**Definition 1.1.1** *The quotient Lie algebra*

$$\text{Shuf}(P) = \text{Sym}(P)/\text{Char}(P),$$

*is called the set of shuffling symmetries of  $P$ .*

Therefore any shuffling symmetry  $S \in \text{Shuf}(P)$  has a unique representative of the form

$$S = f \frac{\partial}{\partial p_0} + \mathcal{D}(f) \frac{\partial}{\partial p_1} + \cdots + \mathcal{D}^k(f) \frac{\partial}{\partial p_k},$$

where  $f$  is a smooth functions of  $(x, p_0, p_1, \dots, p_k)$  and  $\mathcal{D}^i = \mathcal{D}(\mathcal{D}^{i-1})$ , for the reason see [1]. The function  $f$  is called a *generating function* of the symmetry  $S$  and we write  $S_f$  instead of  $S$ . Therefore,  $S_f$  is a shuffling symmetry of the ODE if and only if the generation function  $f$  satisfies the following *Lie equation*:

$$\mathcal{D}^{k+1}(f) - \sum_{i=0}^k \frac{\partial F}{\partial p_i} \mathcal{D}^i(f) = 0.$$

Let us denote by  $\Delta_F : C^\infty(\mathbb{R}^{k+2}) \rightarrow C^\infty(\mathbb{R}^{k+2})$ , the following linear  $k$ -th order scalar differential operator:

$$\Delta_F = \mathcal{D}^{k+1} - \sum_{i=0}^k \frac{\partial F}{\partial p_i} \mathcal{D}^i,$$

which is called the linearization of  $y^{(k+1)} = F(x, y, y', \dots, y^{(k)})$ .

**Theorem 1.1.2** *There exist the isomorphism  $\text{Shuf}(P) \cong \ker \Delta_F$  between solutions of the Lie equation and shuffling symmetries.*

The  $\text{Shuf}(P)$  is a Lie algebra for any distribution  $P$  with respect to the *Poisson-Lie bracket*, which is defined in the following way:

$$[S_f, S_g] := S_{[f, g]} = \sum_{i=0}^k \left( \mathcal{D}^i(f) \frac{\partial g}{\partial p_i} - \mathcal{D}^i(g) \frac{\partial f}{\partial p_i} \right)$$

for any  $f, g \in \ker \Delta_F$ .

**Example 1.1.3** *Functions  $f = a(x, p_0)p_1 + b(x, p_0)$  are generating functions of the vector fields on  $\mathbb{R}^2$  of the form  $b(x, p_0) \frac{\partial}{\partial p_0} - a(x, p_0) \frac{\partial}{\partial x}$ .*

## 1.2 Linear Symmetries

A shuffling symmetry  $S_f$  is called a *linear symmetry*, if the generating function  $f$  is linear in  $p_0, \dots, p_k$ , i.e.,  $f = b_0(x)p_0 + \cdots + b_k(x)p_k$ . With any linear symmetry we associate a linear operator  $\Delta_f = b_0 + \cdots + b_k \partial^k$ , where  $\partial = d/dx$ , and we rewrite the Lie equation for linear symmetries in terms of the algebra of linear differential operators.

**Lemma 1.2.1** *For any linear differential operator  $A = a_0 + \cdots + a_n \partial^n$  and  $L = l_0 + \cdots + l_k \partial^k + \partial^{k+1}$  there are unique differential operators  $C_A$  and  $R_A$  of order  $\leq n - k - 1$  and  $\leq k$  respectively such that  $A = C_A \circ L + R_A$ .*

Here we have a very important theorem:

**Theorem 1.2.2** *[1] A differential operator  $\Delta_f = b_0 + \cdots + b_k \partial^k$  corresponds to a shuffling symmetry  $f = b_0 p_0 + \cdots + b_k p_k$  of the linear differential equation  $L(h) = 0$ , where  $L = A_0 + \cdots + A_k \partial^k + \partial^{k+1}$ , if and only if there is a differential operator  $\nabla_f$  of order  $k$  and such that  $L \circ \Delta_f = \nabla_f \circ L$ . Moreover, the commutator  $[f, g]$  of linear symmetries corresponds to the remainder  $R$  of division  $[\Delta_f, \Delta_g]$  by  $L$ ; that is,  $R_{[\Delta_f, \Delta_g]} = \Delta_{[f, g]}$ .*

Denote by  $\mathfrak{B}(L)$  the Lie algebra of all differential operators  $\Delta$  such that  $L \circ \Delta = \nabla \circ L$ , for some uniquely determined differential operator  $\nabla$ . If  $\text{Sym}(L)$  denote the Lie algebra of linear symmetries of differential operator  $L$ , then we have

**Theorem 1.2.3** 1. *If  $\Delta \in \mathfrak{B}(L)$  then  $R_\Delta \in \text{Sym}(L)$ .*  
 2. *The residue map  $R : \mathfrak{B}(L) \rightarrow \text{Sym}(L)$ , is a Lie algebra homomorphism.*

## 2 Linear Symmetries of Operators

The differential operator

$$L^T = (-1)^{k+1} \partial^{k+1} + \sum_{i=0}^k (-1)^i \partial^i \circ A_i,$$

is said to be *adjoint* to the operator

$$L = \partial^{k+1} + \sum_{i=0}^k A_i \partial^i. \quad (2.1)$$

A differential operator  $L$  is said to be *self-adjoint* if  $L^T = L$  and *skew-adjoint* if  $L^T = -L$ .

The correspondence  $\Delta_f \leftrightarrow \nabla_f^T$  establishes an isomorphism between linear symmetries of the differential equation  $L(h) = 0$  and linear symmetries of the adjoint equation  $L^T(h) = 0$ .

## 2.1 $\mathbb{Z}_2$ - Grading on $\mathfrak{B}(L)$

Let us now assume that  $L$  is self-adjoint or skew-adjoint. Then if  $\Delta \in \mathfrak{B}(L)$  so  $\nabla^T$  does.

using the involution we can decompose  $\mathfrak{B}(L)$  into the direct sum:

$$\mathfrak{B}(L) = \mathfrak{B}_0(L) \oplus \mathfrak{B}_1(L)$$

where

$$\begin{aligned}\mathfrak{B}_0(L) &= \left\{ \Delta : L \circ \Delta = -\Delta^T \circ L \right\}, \\ \mathfrak{B}_1(L) &= \left\{ \Delta : L \circ \Delta = \Delta^T \circ L \right\}.\end{aligned}$$

We will define  $\mathbb{Z}_2$ -parity  $\varepsilon(\Delta) = 0 \in \mathbb{Z}_2$  for  $\Delta \in \mathfrak{B}_0(L)$  and  $\varepsilon(\Delta) = 1 \in \mathbb{Z}_2$  for  $\Delta \in \mathfrak{B}_1(L)$ , and will consider the above decomposition as  $\mathbb{Z}_2$ -grading on  $\mathfrak{B}(L)$ .

**Theorem 2.1.1** *Let  $L$  be a self or skew-adjoint differential operator.*

1. *Then the commutator of operators determines a Lie algebra structure on  $\mathfrak{B}(L)$ , such that*

$$[\Delta_a, \Delta_b] \in \mathfrak{B}_{a+b}(L)$$

*if  $\Delta_a \in \mathfrak{B}_a(L)$ ,  $\Delta_b \in \mathfrak{B}_b(L)$ ,  $a, b \in \mathbb{Z}_2$ .*

2. *Let  $\text{Sym}(L)$  be the Lie algebra of Linear symmetries of operator  $L$ , and  $\text{Sym}_a(L) = R(\mathfrak{B}_a(L))$  for  $a \in \mathbb{Z}_2$  and  $\text{Sym}_b(L) = R(\mathfrak{B}_b(L))$  for  $b \in \mathbb{Z}_2$ . Then  $\text{Sym}(L) = \text{Sym}_0(L) \oplus \text{Sym}_1(L)$  and*

$$[\text{Sym}_a(L), \text{Sym}_b(L)] \subset \text{Sym}_{a+b}(L).$$

We call elements of  $\text{Sym}_0(L)$  by *even symmetries* and elements of  $\text{Sym}_1(L)$  by *odd symmetries* of the equation  $L(h) = 0$ .

## 2.2 Symmetries of Operator $\partial^2 + G(x)\partial + H(x)$

In this part we apply the results to the corresponding operator of 2nd ODE

$$y'' + G(x)y' + H(x), \tag{2.2}$$

It is easy to see that in order two one has only self adjoint operator, thus the following operator is self adjoint and consequently, we will work on

$$\partial^2 + G(x)\partial + H(x), \quad (2.3)$$

instead of equation 2.2. This operator is self adjoint. Therefore the algebra of linear symmetries in  $\mathbb{Z}_2$ -graded.

Let us begin with  $\text{Sym}_0(L)$ . If  $\Delta = A_0 + A_1\partial \in \text{Sym}_0(L)$  then we have  $L \circ \Delta = -\Delta^T \circ L$ . If  $\Delta^T = A_0 - A'_1 - A_1\partial$  then

$$\begin{aligned} L \circ \Delta &= A_1\partial^3 + (A_0 + 2A'_1 + A_1G)\partial^2 \\ &\quad + [2A'_0 + A''_1 + G(A_0 + A'_1) + A_1H]\partial \\ &\quad + A''_0 + GA'_0 + HA_0, \\ \Delta^T \circ L &= -A_1\partial^3 + (A_0 - A'_1 - A_1G)\partial^2 \\ &\quad + [G(A_0 - A'_1) - A_1(G + G')]\partial \\ &\quad + H(A_0 - A'_1) - A_1H'. \end{aligned}$$

Therefore,  $\Delta \in \text{Sym}_0(L)$  implies  $A_0 = -\frac{1}{2}A'_1$  and the function  $A_1 = w$  should satisfy the following differential equation:

$$w''' + (2H - G^2 - 2G')w' + (H' - GG' - G'')w = 0. \quad (2.4)$$

We denote the differential operator corresponding to (2.4) by:

$$\tilde{L} = \partial^3 + (2H - G^2 - 2G')\partial + (H' - GG' - G''). \quad (2.5)$$

If  $\Delta \in \text{Sym}_1(L)$  then  $L \circ \Delta = \Delta^T \circ L$  and we obtain  $A_1 = 0$  and  $A'_0 = 0$ . Thus  $\Delta \in \text{Sym}_1(L)$  is and only if  $\Delta$  proportional to the identity operator. Finally we have the following theorem which is a generalization of the theorem 2.5.1 of [1].

**Theorem 2.2.1** *The Lie algebra of linear symmetries of the differential operator (2.5) has the following description:*

1.  $\text{Sym}_0(L) = \left\{ -\frac{1}{2}w' + w\partial : \tilde{L}(w) = 0 \right\}.$
2.  $\text{Sym}_1(L) = \mathbb{R}.$

**Example 2.2.2** *Let us suppose that in (2.3) we have  $G(x) = 0$ . Thus the new operator is called **Schrödinger operator**. It is possible to see that the even symmetries of Schrödinger operator is isomorphic to the Lie algebra  $\mathfrak{sl}(2)$ . And of course the Schrödinger operator does not have any nontrivial odd symmetries.*

## References

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